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Complete chaos synchronization of two identical Lorenz systems *using nonlinear active control*

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Abstract:

Complete synchronization scheme was explored in this work. In this approach we make use of the nonlinear active controller to ensure the synchronization between the master and slave systems, with identical parameters and initial conditions, to ensure reliable and stable state of the synchronization. The approach demonstrates a robust process, as the active controller being designed ensures the two identical systems synchronize over time. Tested against mismatched initial conditions we were able to verify the sensitivity of the Lorenz system towards initial conditions, the phase portrait and butterfly effect of the system was also illustrated to affirm the validity of this claim, the method demonstrate resilience towards mismatching initial conditions, giving room to areas of possible application such as securing communication, signal processing and control theory.

Keywords: synchronization, active control, nonlinear dynamics, attractors, butterfly effect.

Introduction

In today's world, digital communication has become an integral part of every society, a large amount of data is being shared every day, be it personal information, transactions in finance, or the sharing of confidential messages. Due to such information circulation, individuals, with governmental and non-governmental organizations face severe threats from unauthorized par- ties that intercept the information unlawfully, putting the privacy of the parties concerned in jeopardy. Conventional techniques or algorithms such as hashing, blowfish, and so on are being cracked with the help of quantum algorithms [3].

1.1 Chaos and Dynamical Systems

1.1.1 Nonlinear Dynamics of a System

The dynamics of any given system are significantly impacted by the non- linearity of the terms in the systems. Nonlinear equations, as opposed to linear equations, permit for discrete solutions for a periodic oscillations in the system specifically concerning equilibrium. This makes solving nonlinear equations much more difficult analytically, otherwise almost impossible without the use of numerical or semi-analytical methods. Therefore, to achieve approximate answers, numerical methods are used, furthermore, geometrical patterns and reasoning are shown to be highly successful in obtaining a qualitative knowledge of the behavior of solutions[4]. Differential equations (DEs) have to do with the evolution of dynamical systems in a time-continuous domain. This so-called evolution is expressed by:

$$x' = f(x) \qquad (1)$$

Where f denotes the function of flow while x is the flow variable.

1.1.2 Phase Space and criteria of stability

A dynamical system is defined and further described by the process of its own evolution (differential equations(DEs) and difference equations), ex- plains how a given point x changes within a phase space with respect to the time in the system denoted by ”t”, and its phase-space (the possible collection of all states of x of the system). Here, we’ll use an arbitrary continuous dynamical system as an example to help grasp the basic concept. A general dynamical system can fully be written as a set of linear and nonlinear ordinary differential equations. let us expand upon the previous equation given in Eq. (1)

$$\dot{x}_1 = f_1(x_1, x_2, x_3, \dots, x_n), \dots \dot{x}_n = f_n(x_1, x_2, x_3, \dots, x_n).$$

Since the general form of any dynamical system’s phase space consists of x-coordinates $x1, x2, x3, \dots, xn$, we can recall back to Eq (1) as a usual n- dimensional system of differential equation or an nth-order system due to its intrinsic n-dimensionality. Subsequently, we know n represents the dimension of the phase space of the system. In the phase space being studied, solutions of the system $(x1, x2, \dots, xn)$ in particular is corresponding to some point traveling or moving on a curve within the system’s plane; this is called a trajectory, and a phase portrait is the representative collection of the tra- jectories along which the points move about. We can claim that the entirety of the phase space is full with trajectories and points moving around the trajectories, as each point in the space can be used as the initial condition necessary for the dynamical system. The equilibrium (or constant) solutions, or critical (or fixed) points, are those for points for which f(x) in Eq (1) is zero. The stability of the system is defined based on the Lyapunov functions negative definite. given a chosen $k > 0$.

1.1.5 Chaos with Lyapunov Exponent

Chaos can also be identified by Lyapunov exponents.[5] Based on its value, the Lyapunov exponent is a systematic technique that can be used to assess how a dynamical system behaves in a chaotic or non-chaotic mannerism. Regarding parameter fluctuations (constantly increasing or decreasing), the vast majority of dynamical systems studied typically display the following types of solutions: stable or otherwise fixed, periodic and sometimes quasi- periodic, chaotic (unpredictable), unstable or typically divergent. In one- dimensional systems, chaotic behavior is usually indicated by a positive Lyapunov’s exponent value, whereas periodic orbit behavior is specified by a negative Lyapunov’s exponent value, and marginally stable orbit is indicated by a zero Lyapunov’s exponent value [10].

1.2 Aim and Objectives

We aim to come up with a safe communication system by utilizing the chaotic nature of a Lorenz system, with an emphasis on studying encryption and decryption processes for a better protection against cyber attacks.

OBJECTIVES

• To analyze the theoretical aspects of Lorenz system and it’s chaotic properties for application for encryption and decryption.

• To design an encryption-decryption framework utilizing complete chaos synchronization scheme with active control with identical Lorenz systems.

Conclusion

Numerous studies have explored chaotic system synchronization and its potential for secure communication. Despite these advances, **significant research gaps remain** particularly in developing more **robust, noise-resilient, and adaptable encryption schemes** that can withstand real-world variability and cyber threats. This literature review highlights key contributions, limitations, and the evolving landscape of chaos-based secure communication systems.

Over the past few decades, chaotic systems have been widely applied in diverse domains, including finance, signal processing, epidemiological modeling, and control systems. One key application area is secure communication, where synchronization of chaotic states between a transmitter and a receiver enables effective encryption and decryption. Chaotic synchronization allows these systems to securely transmit data, capitalizing on the sensitive dependence on initial conditions to prevent unauthorized access.

Recent research efforts have focused on enhancing synchronization strategies and encryption efficiency. One study demonstrated that chaotic synchronization can significantly reduce the impact of measurement noise by using controllers to secure communication. Their results showed improved robustness and message recovery even under high-noise conditions [2].

Another notable contribution introduced a **fractional-order chaotic system with hidden attractors** [13]. Fractional calculus adds complexity to system dynamics, making them more unpredictable and secure. These systems are harder to intercept and synchronize without precise knowledge of their parameters, thereby strengthening encryption capabilities.

In [8], the authors introduced an innovative fusion of **brain emotional fuzzy control (BEFCC)** and chaos synchronization for **audio encryption and decryption**. This novel integration of emotional intelligence and chaotic systems expands the scope of secure communication beyond traditional text and image data, paving the way for **secure multimedia transmissions**.

Further, a PID controller coupled with a quasi-sliding mode controller (QSMC) was used to synchronize master-slave Lorenz systems for **IoT-based encrypted environmental signal transmission** [6]. The study highlighted the practical implementation of chaos-based security in sensing nodes, although the approach demands high precision in maintaining system synchronization.

Another approach utilized the **Sprott master-slave chaotic system** with a sliding mode controller designed using a Lyapunov-based method to ensure stability and robustness [14]. The simulation results demonstrated that synchronization error converges to zero over time, indicating complete synchronization even in the presence of system disturbances.

The **Lorenz system**, originally designed for meteorological modeling, has also been widely adopted in secure communication systems. By synchronizing two Lorenz systems (master and slave), secure data transmission is achieved through shared chaotic dynamics [7]. Modern research in this area emphasizes designing robust controllers—such as **active, adaptive, and sliding mode controllers**—to maintain synchronization regardless of noise, perturbations, or parameter mismatches [11].

Adaptive control techniques, in particular, adjust system parameters dynamically to counter real-world uncertainties, enhancing synchronization robustness and reliability [12]. These controllers are often grounded in **stability theory** to ensure phase and amplitude synchronization while minimizing synchronization error.

Additionally, encrypted messages are often **embedded using chaotic masking**, where the message is obscured within a chaotic signal and only recoverable by an identical system with matched parameters [1]. This approach offers high resistance to interception, as even minor parameter deviations can prevent successful decryption.

Research gap

Despite the promising outcomes of prior studies, several challenges persist:

1. Many synchronization schemes struggle to maintain **robustness under high noise levels** and **parameter mismatches**, especially in real-time applications.
2. Existing approaches often **lack adaptability** to changing external conditions, limiting their practical deployment.
3. The **convergence behavior** and **performance benchmarks** of these schemes are not comprehensively analyzed or standardized across studies.
4. Most studies focus on specific applications (e.g., text or audio encryption), but **broader evaluations** for secure multimedia, IoT, or critical infrastructure communication remain under explored.
5. There is a need for **simplified yet effective control schemes** that reduce computational complexity while maintaining synchronization accuracy.

Current work

In this study, we propose and implement a nonlinear active control scheme to achieve **complete synchronization between two identical Lorenz chaotic systems**. Although the systems share identical structural dynamics, they are initialized with different state conditions, allowing us to evaluate the controller's performance under parameter mismatches. A control input is designed based on **Lyapunov stability theory**, ensuring the synchronization error converges asymptotically to zero. Theoretical analysis is supported by **numerical simulations**, which demonstrate the effectiveness and robustness of the proposed method. The results confirm stable synchronization, even in the presence of initial mismatches, thereby underscoring the method's **viability for real-world applications** such as **secure communication, signal processing, and control systems**. The inherent **chaotic properties of the Lorenz system**, particularly its sensitivity to initial conditions, further enhance the potential of this approach for encryption and cyber-m defense mechanisms.

Methodology

Let the master system be given as

$$x' = Ax + \phi(x) \qquad (2)$$

be the general form of the master system, with $x = (x1, x2 \ldots xn)^T$ as the state vector variable. In this system A is the linear part while

$$\phi : Rn \rightarrow Rn$$

is considered as nonlinear part of the system. Consider a slave (a Lorenz system)system that is identical to the master system added with a control function.

$$y' = By + \psi(y) + u(x, y) \qquad (3)$$

In this system $y = (y1, y2 \ldots yn)^T$ is the state vector, while B is the linear part of the system, $\psi : Rn \rightarrow Rn$ is the nonlinear part. $u(x, y)$ is the nonlinear control to be designed that will ensure the slave system synchronize with the master system overtime.

Since the master and slave system are identical then both linear and non- linear part of the system are equal i.e. A = B and $\phi = \psi$, where x and y served as the state vectors of the two identical systems. Now, defining the synchronization error, $e(t)$ as

$$e(t) = y(t) \pm x(t)$$

The operation in this error to be used depends on the synchronization scheme under consideration. For stability the Lyapunov function can be defined as

$$V(e) = \frac{1}{2}(e^T e)$$

If the function V (e) is positive definite, then with a suitable controller u(x,y), we can have that the function derivative V' is negative definite. Thus by Lyapunov stability criteria, synchronization is achieved between the two system.

Complete Synchronization Scheme

In this scheme the difference of the state-vectors of the two Lorenz systems synchronized tends to zero overtime. The master system is said to be in complete synchronization with the slave system if

$$\lim_{t\to\infty}\|e(t)\| = \lim_{t\to\infty}\|y(t) - x(t)\| = 0$$

Where $\|.\|$ is considered as the euclidean norm. Assuming the synchronization error is e $e = y_i - x_i, i = 1,2,3.$ the synchronization error dynamics is then taken as

$$e' = By + \psi(y) + u(x,y) - Ax - \phi(x)$$

Clearly the master and slave system will achieve complete synchronization if $u(x,y)$ is chosen as:

$$u(x,y) = -By - \psi(y) + Ax + \phi(x) - ke$$

Where k is the control gain.

Analysis, Result and Discussion

(Master-Slave System with necessary parameters)

Consider the Master System

$$x_1' = \alpha(x_2 - x_1)$$
$$x_2' = \beta x_1 - x_2 - x_1x_3$$
$$x_3' = -\gamma x_3 + x_1x_2$$

Where, x_1, x_2, x_3 are the state variables and α, β, γ are the parameters.

Slave System

$$y_1' = \alpha(y_2 - y_1) + u_1$$
$$y_2' = \beta y_1 - y_2 - y_1y_3 + u_2$$
$$y_3' = -\gamma y_3 + y_1y_2 + u_3$$

We added $u1, u2, u3$(control inputs) which are going to be designed as control laws that will enable the slave system to synchronize with the master system. These controllers enable the systems to behave in a matching trajectory in identical manner overtime. In order to synchronized the systems, we defined the error vector $e(t) = (e_1(t), e_2(t), e_3(t))^T$by

$$e_i = y_i - x_i, \text{then for } i = 1,2,3$$
$$e_1 = y_1 - x_1$$
$$e_2 = y_2 - x_2$$
$$e_3 = y_3 - x_3$$

To achieve complete synchronization, $ei \rightarrow 0$ for i = 1, 2, 3. This implies that yi $\rightarrow$ xi in all instances, thus the master-slave system synchronizes. Substituting the given values of the derivatives in the master-slave model into the error dynamics e', we have;

$$e_1' = y_1' - x_1'$$
$$= \alpha(y_2 - y_1) + u_1 - [\alpha(x_2 - x_1)]$$
$$e_1' = \alpha(y_2 - y_1) + u_1$$

Consider e_2:

$$e_2' = y_2' - x_2'$$
$$= \beta y_1 - y_2 - y_1y_3 + u_2 - \beta x_1 + x_2 + x_1x_3$$
$$= \beta(y_1 - x_1) - (y_2 - x_2) - (y_1y_3 - x_1x_3)$$
$$\therefore e_2' = \beta e_1 - e_2 - (y_1y_3 - x_1x_3) + u_2$$

Similarly for e_3:

$$e_3' = \gamma e_3 + (y_1y_2 - x_1x_2) + u_3$$

Now we need to design the control inputs $u1, u2, u3$ that will make all the errors converge to zero i.e ei $\rightarrow$ 0, for all i. For easier set up lets set the inputs to be opposite of the nonlinear terms of the system, this will bring about negative feedback for all the errors; this choice will make the function to be negative definite.

First input(control):

To impose a negative feedback on the synchronization errors i.e. for $e_1' = -k_1e_1$ where $k_1 > 0$ (constant), from error dynamics:

$$e_1' = \alpha(e_2 - e_1) + u_1$$
$$\text{Choose } u_1 = -\alpha(e_2 - e_1) - k_1e_1$$
$$\Rightarrow e_1' = \alpha(e_2 - e_1) + (-\alpha(e_2 - e_1) - k_1e_1)$$
$$= \alpha(e_2 - e_1) - \alpha(e_2 - e_1) - k_1e_1$$
$$= -k_1e_1.$$

Second input (control):

For $k_2 > 0$, the error dynamics:

$$e_2' = \beta e_1 - e_2 - (y_1y_3 - x_1x_3) + u_2$$
$$\text{Choosing } u_2 = -\beta e_1 + e_2 + (y_1y_3 - x_1x_3) - k_2e_2$$

so that we have $e_2' = -k_2e_2$.

Third input (control):

Following the same identical procedure as in the first two control inputs of the system, we want:

$$e_3' = -k_3e_3 \text{for } k > 0$$
$$\Rightarrow e_3' = \gamma e_3 + (y_1y_2 - x_1x_2) + u_3$$
$$\text{Choosing } u_3 = -\gamma e_3 - (y_1y_2 - x_1x_2) - k_3e_3$$
$$\text{so that } e_3' = -k_3e_3 \text{ for } k_3 > 0$$

The error dynamics are reduced to:

$$\begin{cases} e_1' = -k_1 e_1 \\ e_2' = -k_2 e_2 \\ e_3' = -k_3 e_3 \end{cases}$$

For stability, we define the Lyapunov function $V(e)$ as

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2)$$

differentiating,

$$\dot{V}(e) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3$$

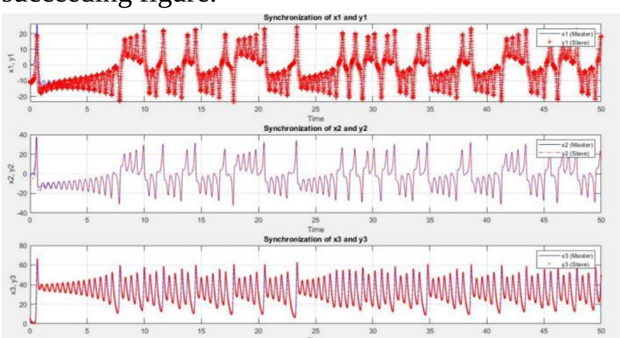
Putting values of $\dot{e}_i$ for $i = 1,2,3$:

$$V' = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2, \text{ for } k_i > 0$$

This function is a negative definite. Hence, we say that by Lyapunov Theory of Stability, the dynamics are asymptotically stable and converge exponentially to zero, thus complete synchronization is achieved.

Example (Numerical simulation):

Let us use simulation (numerical) to systematically verify the result of the master-slave chaotic synchronization. Use initial conditions $x0 = (x1, x2, x3) = (3,-4,2)$ for the master system and $y0 = (y1, y2, y3) = (-10,-11,5)$ for the slave system, the error vector conditions as $ei = (y0 - x0)$. Suppose that k has an invariant value i.e. k1 = k2 = k3 = 1 (chosen). After applying the controllers, the state variable’s time analysis is presented in the succeeding figure.



The graph shows how the state variables of the identical master-slave systems synchronized (complete synchronization) over time. The first subplot represents the state variables (x1 and y1) , the second subplot shows (x2 and y2), and the third subplot displays (x3 and y3).

The first subplot shows the analysis of time series of the synchronized state of the first chaotic variables for both the master-slave system. The second subplot shows the analysis of time series of the second corresponding chaotic variables between the two identical systems. Likewise the third subplot is the series analysis of the third corresponding variable between the two identical systems.

Conclusion

This research article demonstrated the successful implementation of complete synchronization between two identical systems using active control. The systems shared identical linear and nonlinear components, differing only in their state variables. Active control was employed to achieve synchronization over time, and Lyapunov stability theory was applied to verify the stability of the synchronized state. The results confirmed effective synchronization, and the potential application of this method in secure communication was discussed in the literature review. Furthermore, the method's robustness was validated under parameter mismatches, underscoring its viability for real-world applications. This approach offers promising potential in secure and efficient communication systems, contributing a novel strategy for countering cyber attacks.

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