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Enhancing Geometric Understanding Through Reflective Practice

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Abstract:

In this article, we argue that folding back is successful when the learners engage in exploratory talk. To support our argument, we sourced data from a Grade 10 mathematics classroom of 54 learners who participated in a four-week teaching experiment conducted by the second author. We mainly focused on talks in two groups of learners to address the silence of literature on folding back that alludes to the kind of talk that learners engage in. Data were captured through video recording of learners' interactions as they worked on the tasks in different sessions. We present these data as transcribed extracts of talks that the learners held and synthesise them into stories through Polkinghorne's narrative mode of data analysis, also using a process that Kim referred to as narrative smoothing. Pirie and Kieren's conception of folding back and Wegerif and Mercer's three ways of talking and thinking among learners were used as a heuristic device for synthesising the stories. The narratives illustrate that exploratory talk promotes folding back, where learners build on each other's ideas to develop geometry understanding. Therefore, the significance of this article is that for classrooms that wish to promote growth in understanding through folding back, the type of talk that should be normative is exploratory talk.

Keywords: folding back, dynamical theory of the growth of mathematical understanding, dialogical framework for researching peer talk, geometry understanding.

Introduction

The study of high school geometry remains a challenge in mathematics education. However, some studies show that folding back in Pirie and Kieren's theory of growth in mathematical understanding is beneficial for growth in geometry understanding. For example, when Gülkilik et al. (2015) examined Grade 10 learners' understanding of geometric transformation, they recommended that learners should be encouraged to fold back to the inner levels to strengthen how concepts are understood. In addition, Pirie and Kieren's theory enabled Yao (2020) to capture the evolving understanding of geometric transformation when working with secondary school preservice teachers. Also, a study by Akarsu (2022) showed that the theory, when used together with Van Hiele's model of geometric thinking, offers a unique perspective for appreciating how learners develop an understanding of geometric reflection. While framing their study on Piere and Kieren's theory jointly with Duval's (2006) Semiotic Representation Theory, Gülkılık et al. (2020) asserted that the former theory enabled the construction of images about geometric concepts as well as noticing mathematically significant and traceable properties of that concept. However, while some researchers have used Pirie and Kieren's theory and its extension by Martin (2008) to track and trace learners'

understandings (Yao, 2021), there seems to be no study that examined the kind of talk that takes place during these processes. Yao and Manouchehri (2022) argue that work that focuses on learners' growth of geometry understanding is still scarce. They further claim that how folding back might occur is not yet clear. Factors that enhance folding back towards developing specific mathematical actions are not explicit. For example, Chiphambo and Feza (2020) conducted a study with Grade 8 learners and found that teaching methods influenced how to succeed in geometry. The specific methods mentioned are the hands-on and mind-on learning of geometric concepts. However, studies on folding back and focusing on learners' understanding of geometry do not analyse the type of talk learners engage in so that understanding can occur. This silence in the literature occurs regardless of acknowledgements of enhanced cognitive activity that come with talk interventions in classrooms (Soysal, 2019). Though Yao and Manouchehri (2022) detailed how folding back takes place, they also did not classify the type of talk that the learners use. Therefore, in a context where folding back offers learners a way of deepening learning through visiting earlier forms of understanding (Hähkiöniemi et al., 2023) to build deeper mathematical understanding and language (Corovic, 2022), this article argues that folding back is successful when the learners engage in exploratory talk. Potentially, 'exploratory talk provides the richest and most valuable contribution to the quality of learning' (Chan, 2020, p. 10). In a South African study by Hardman and Lilley (2023), exploratory talk among learners symptomised a 'truly dialogical interaction, where partners together construct meaningful [mathematics] knowledge through negotiation and debate'. (p. 13). In mathematics, Sfard (2008) brought to the fore the relationship between thinking and speech in her theory on commognition, a term she coined to combine thinking with communication. Mercer (2008) has also drawn a correlation between talking, reasoning and understanding in education. He classified such talk into three types: disputational, cumulative, and exploratory (2008). Disputational talk is observable when there are no agreements and individualised decision-making. In cumulative talk, learners construct knowledge common among themselves through repetition, confirmation and elaborations. In exploratory talk, 'partners engage critically but constructively with each other's ideas' (Mercer & Littleton, 2007, p. 72). For example, a Flemish study conducted by T'Sas (2018) found that when exploratory talk is taught properly, the learners learn from each other to improve group reasoning. This was not the only benefit; the learners improved skills on argumentation and problem-solving. A South African study conducted in a Physical Science classroom showed that learners used exploratory talk when working on tasks requiring application (Radebe & Mushayikwa, 2023). In addition to the benefits of exploratory talk in classrooms, it encouraged the development of critical thinking, the elements of which were in expressing differing views and offering clear justifications (Liang & Fung, 2021). These benefits can be extended to mathematics learners as evidenced in a South African study by Webb et al. (2017). Furthermore, '[i]n [exploratory] talk, all partners actively participate, opinions are sought, and decisions are jointly made' (Røsseland et al., 2022, p. 1). However, the occurrence of exploratory talk in folding back as a process through which understanding develops has not been captured in literature. In this study, the following research question was addressed: What kind of talk is necessary for folding back to help learners develop geometry understanding?

Theoretical Framework

To analyse learners' growth in understanding geometry, we adapted Pirie and Kieren's (1994) dynamical theory of the growth of mathematical understanding. This theory contains eight nested layers that explain the non-linear process of learning mathematics. Table 1 lists these layers and their descriptions. Pirie and Kieren's (1994) theory has a vital characteristic called folding back. Through folding back, learners reconstruct and elaborate on an inner level of understanding to support the next level of understanding. Learners use folding back when they fail to apply their understanding at a specific level. Instead, they move back to an inner level to extend their understanding and reorganise it so that they can address their failure (Gülkılık et al., 2015). Martin (2008) argued that the notion of folding back was initially underdeveloped and unelaborated. Subsequently, he proposed a framework that elaborated on folding back as having three tenets. The three tenets are the source of intervention, the form of folding back and the outcome of folding back. In this article, folding back is a lens through which we explored how learners build on each other's ideas to develop their understanding of geometry. The source of intervention invokes a learner to fold back. The source of intervention may be the self, a peer, a teacher or material at a learner's disposal (Martin, 2008). According to Martin (2008), the form of folding back describes actions that learners engage in because of the source of intervention. These actions may involve working at an inner layer using existing understanding, collecting at an inner layer, moving out of topic, working there, and causing a discontinuity. The actions that make up the forms of folding back may

TABLE 1: The eight layers of the dynamical growth of mathematical understanding.

Level of mathematical understanding	Layer	Description
Informal	Primitive knowing	A first innermost layer where learners bring what they know and can do in their learning contexts (Pierie & Kieren, 1994)
	Image making	A layer where learners begin to conceptualise ideas based on images from their primitive knowing (Cobb, 2012)
	Image having	A layer where learners recognise general properties of the created mathematical images (Gibbons, 2012)
	Property noticing	Where learners reflect, recognising attributes and features of mental images (Yao & Manouchehri, 2022)
Formal	Formalising	A layer in which learners are able to work with generalised properties without reference to a specific image or action (Amin & Sulaiman, 2021)
	Observing	A layer where learners coordinate mathematical theorems and concepts to solve the problem (Mardiana, Susiswo & Hidayanto, 2017)
	Structuring	A layer where learners focus on determining the relationships between theorems and prove them using formal argumentation (Mardiana, Susiswo & Hidayanto, 2017)
	Inventising	The outer layer where learners have a complete understanding of structures about mathematical concept and can pose questions that lead to a 'new concept' (Mardiana, Susiswo & Hidayanto, 2017, p. 36)

Note: Please see the full reference list of the article for more information.

Result in either effective or ineffective folding back. Effective folding back occurs when learners apply a comprehensive understanding to solve the initial problem (Martin, 2008). Contrary to that, ineffective folding happens when learners cannot apply comprehensive understanding to the initial problem. Furthermore, we had to analyse how learners build on each other's ideas as peers. We found Wegerif and Mercer's (1997) dialogical framework for researching peer talk in which they define social modes of thinking through three kinds of talk relevant for our analysis. Wegerif and Mercer's perspective on talk as a thinking tool enabled them to delineate thinking that is 'embodied in different types of talk' (p. 60). They found these kinds of talk appropriate to explain how to use talk to build on each other's ideas. These talks are disputational talk, cumulative talk and exploratory talk (Wegerif & Mercer, 1997). In disputational talk, participants tend to be defensive in justifying their contribution to the talk because others may view their ideas narrowly (Mercer, 2008). Mercer (2008) characterises cumulative talk as repetitions, confirmations and elaborations that avoid differing opinions to maintain the image of a group. However, the partners offer statements and suggestions to consider jointly in exploratory talk (Wegerif & Mercer, 1997). They may challenge and counter-challenge suggestions, but challenges are justified, and they can offer alternative hypotheses. All partners participate actively, and opinions are sought and considered before making joint decisions. 'Compared with the other two types of talk, in exploratory talk is more publicly accountable and reasoning is more visible in the talk' (Mercer & Wegerif, 2004, p. 87). Furthermore, exploratory talk fosters critical thinking and cognitive development (Mercer, 1996). Although not an exhaustive list, the following features were identified by Mercer (2008) to characterise exploratory talk:

- Everyone is encouraged to contribute.
- Everyone listens actively.
- People ask questions.
- People share relevant information.
- Ideas and opinions are treated with respect.
- There is an atmosphere of trust.
- There is a sense of shared purpose.
- Contributions build on what has gone before.
- People give reasons for their thinking.
- Ideas may be challenged.
- The group seeks agreement for joint decisions.

The three types of talk are 'not meant to be descriptive categories into which all observed speech can be neatly and separately coded' (Wegerif & Mercer, 1997, p. 54). Both the Pirie and Kieren (1994) dynamical theory of the growth of mathematical understanding and Wegerif and Mercer's (1997) dialogical framework for researching peer talk are not meant to chunk data into fixed categories but will be used to explicate the data and illustrate that exploratory talk is essential for folding back to occur.

Methods

We sourced data from a Grade 10 mathematics classroom of 54 learners who participated in a four-week teaching experiment conducted by the second author. This article mainly focuses on talks in two groups of learners. Teaching experiments are a sequence of teaching episodes through which the researchers study and theorise about learners' learning and reasoning in mathematics (Lamb & Geiger, 2012). In addition, teaching experiments offer researchers opportunities to learn the mathematical knowledge of learners and how they construct it through talk (Steffe &

Thompson, 2000). Through teaching experiments, researchers can systematically investigate the development of learners' meaning (Moore et al., 2019). Hence, the teaching experiment allowed us to identify critical incidents where learners' exploratory talk allowed them to fold back to grow their understanding as they worked through some geometry tasks. We did not expect these incidents to happen, but they have 'the potential to deepen our understanding of learners' [growth in mathematical understanding]' (Choy, 2014, p. 143). The incidents we analysed for this study came from two groups of learners out of 54 Grade 10 mathematics learners. In the first group, the participants' pseudonyms were John and Koena; in the second group, they were Lesiba, Lebogang and Sipho. The sample size of these two groups is consistent with the teaching experiment research design as it allows one or a few learners as participants in the study (Steffe & Thompson, 2000).

Data were captured through video recordings of learners' interactions as they worked on the tasks in different sessions. We present data as extracts of transcripts of talks that the learners held because talk is used to check learners' growth in understanding. We synthesised the extracts into stories through the narrative mode of data analysis (Polkinghorne, 1995). This mode of data analysis allowed us to fill gaps in narratives and bring meaning that was not necessarily explicit in the extract. Kim (2016) refers to this process of filling gaps in data as narrative smoothing. Narrative smoothing helps make narratives coherent and engaging. In narrating the extracts, the frameworks by Pirie and Kieren's (1994) conception of folding back, Martin's (2008) elaboration of folding back and Wegerif and Mercer's (1997) three ways of talking and thinking among learners were used as a heuristic device for synthesising the talks. We chose the extracts of talk where understanding of geometric ideas can be explained through folding back to illustrate instances of exploratory talk.

Quality criteria

Teaching experiments are judged on their ability to provide a model or argument related to teaching and learning because they are grounds for researchers to learn (Molina et al., 2007). Hence, this article argues that folding back is successful when the learners engage in exploratory talk. The argument is not limited to one extract but to two and is new to studies on folding back, as shown in the introduction of the article. However, the argument cannot be replicated in other settings as this is not a requirement for teaching experiments (Steffe & Thompson, 2000). Instead, it can be transferred to other settings for progressive refinement of the argument by either the authors of this article or other researchers interested in folding back. Retrospective analysis of the data for this article was made possible by capturing data on video to enable both the first author and the other authors to recollect what happened during data collection.

Ethical considerations

The second author applied for ethical clearance to conduct the study through the university's ethical committee, the school's principal, and the parents of participating learners. Ethical clearance to conduct this study was obtained from the University of Limpopo, Turfloop Research Ethics Committee (Clearance Certificate no. TREC/51/2016:PG). He also informed the learners who willingly participated in data construction without compensation for the study. As participants, the learners had a chance to enrich their knowledge of geometry as the study stayed within the prescribed curriculum.

Results

The analysis presented here illustrates instances of talks where the process of folding back happened as learners engaged in

exploratory talk as they worked on two geometry learning tasks. The first task assessed learners' ability to apply the theorem that states that an exterior angle of a triangle is equal to the sum of its two opposite non-adjacent interior angles. The task (Figure 1) was stated as follows. If ∠A = 90°, prove that DÔC = 45°. Using layers of understanding by Pirie and Kieren (1994) as a referent, the assessment task was pitched at the property noticing level. Learners had to notice that they had to apply properties of exterior angles of a triangle and the sum of angles in a right-angled triangle. The following extract starts after John asked the class teacher for help:

1.1. John: Sir, please help us to find DÔC.

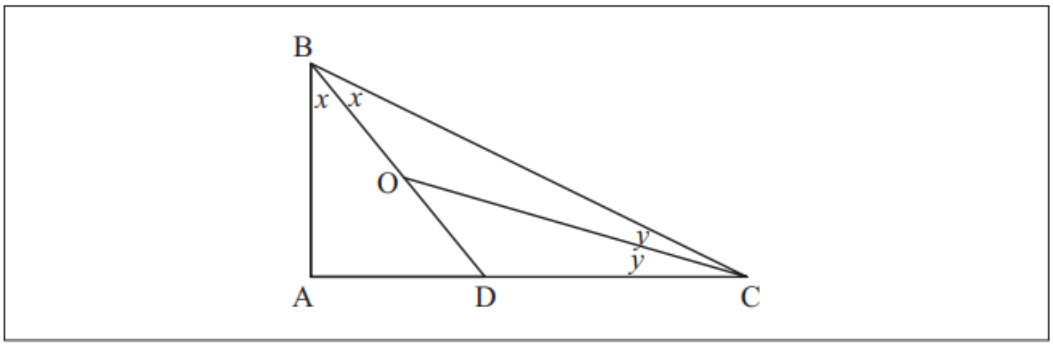
1.2. Teacher: What will be the sum of ∠B?

1.3. John: 2 x … hmmmm ∠B1 is equal to ∠B2 and ∠B2 is equal to x, which means each of ∠B1 and ∠B2 is x, do you understand?

1.4. Koena: Eish … [shaking head-indicating that he does not understand].

1.5. John: It means that here [pointing at ∠B and ∠C] it is 2x and 2y …

1.6. Koena: Oh, I understand now, meaning ∠B is having two angles …



Source: Mabotja, K.S. (2017). An exploration of folding back in improving Grade 10 students' reasoning in geometry. Unpublished master's thesis, University of Limpopo, Mankweng, South Africa (p. 63). Retrieved from http://hdl.handle.net/10386/1805

FIGURE 1: An exterior angle of a triangle equals the sum of its two opposite non-adjacent interior angles.

The teacher intervened by asking a question that looked for the sum of the angles that formed ∠B. John's answer suggested that he understood the representation of these angles and then showed that they are equal. Hence, he gave the sum of the angles to be 2x. In this instance, the teacher's intervention was explicit and intentional as it led John to the desired response. In a similar manner, John realised that ∠C = 2y. In so doing John was able to fold back from the property noticing level to the image having level. John then checked with Koena if he shared his understanding; this comparison is one of the indicators of exploratory talk. However, Koena's reaction on line 1.4 suggested that he did not understand. John drew Koena's attention to notice that ∠B and ∠C were sums of 2x and 2y. This part of exploratory talk led Koena to move from primitive knowing to image having. As observed by Mercer et al. (1999, p. 97), 'in exploratory talk, knowledge is made publicly accountable and reasoning is visible in the talk'.

When John shared his ideas with Koena, the ideas contributed to Koena's development of geometry ideas. Koena was observed building on John's ideas and reformulating them as: '∠B is having two angles ...' (line 1.6). Folding back by both John and Koena (moving from image having level to property noticing) resulted in an environment where the two learners could develop an idea together. In this instance, we observed John's exploratory talk characterised by mathematical actions such as explaining and justifying ideas to his peer. Thus, learners'

ability to explain their own ideas become enhanced when they engage in exploratory talk.

The talk between John and Koena continued in the extract that follows. It began with John questioning how they can evaluate ∠ x and ∠ y. This makes the teacher the source of the intervention. Again his intervention was intentional and explicit because it ultimately led the learners to the size of ∠ x and ∠ y:

1.7. John: Yes … question is, how are we going to calculate x and y?

1.8. Teacher: What will be the sum of angles in that triangle?

1.9. John: 90° + 2x + 2y = 180° [talking and writing]

1.10. John: 90° + 2x + 2y = 180° … 2x + 2y = 90°

1.11. Koena: So, do we find x first?

1.12. John: What if we divide by 2?

1.13. Koena: We divide 2x + 2y by 2 … then we remain with x + y = 45°

In this extract, it can be claimed that when John asked the question on how to calculate the sizes of x and y, he trusted that the group members would share relevant information. In sharing the responsibility of working out the problem, the teacher asked them about the sum of angles of a triangle. The teacher's questioning guided learners to knowledge that they had met before. Hence, the response by John, who talked as he wrote: 90° + 2x + 2y = 180°. In so doing, the teacher invoked John to collect knowledge at an inner level. In this case, John's mathematical actions suggested that he noticed the properties of the sum of angles in a triangle. These actions were evidence that he was at a property noticing level. The talk proceeded with Koena and suggested that they solve for x first. Instead, John suggested that they divide by 2. Koena proceeded to divide the terms of the equation by 2 to get x + y = 45°. This move indicated that there was no competition of ideas in the group. Instead, there was a sense of shared purpose.

The next extract shows the interactions between learners when they finalised their work on the task. The extract began with an intervention from John. He suggested that the other learners should rotate the drawing and redraw it (Figure 2):

1.14. John: Let us say we put it this way [rotate the learning activity drawing] … draw it [suggest that other peers draw].

1.15. Sipho: How?

1.16. John: Looking at it the way it is ...

1.17. Sipho: I am going to make mistake.

1.18. John: [Starts drawing] [see Figure 2]

1.19. Koena: Ohoo ∠0 is an exterior angle.

1.20. John: Yes, my friend … so ∠0 is equal to B2 + C2 (exterior) … ohooo … wooowwww! ∠0 is equal to ∠B, ∠B is the same as, look at it x + y; ∠0 = x + y … ∠0 = 45°.

The talk started with John encouraging the group members to redraw the diagram after he rotated it. It seems that John wanted to rotate the diagram so that it became easier to identify properties of the diagram that would help them solve the problem at hand. The dynamical theory of the growth of mathematical understanding refers to this as the property noticing layer. He drew the learners' attention to the triangle BOC where the angle BÔC is located. The learners had to move out of the idea that was required on the task to transformation of the diagram. Immediately, John drew the diagram that isolated the required and sufficient information to finalise the proof. Koena was able to notice that DÔC is an exterior angle to Δ BOC. John extended this observation to conclude that DÔC equals x + y = 45°. In the extract John referred to angles x and y as B_2 and C_2.

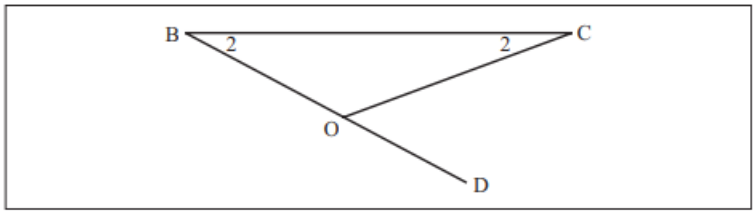


FIGURE 2: Rotation of a part of Figure 1 to clarify its properties.

John's utterances showed that he was impressed by the observation by Koena. He said 'exterior … ohooo … wooowwww!'. This utterance can be classified as a characteristic of exploratory talk where learners treat each other's ideas and opinions with respect and appreciation. Again it can be argued that through folding back, an environment in which learning results from co-construction of ideas and extension of these ideas is able to flourish.

After the class was taught about the midpoint theorem, the learners were assessed using the task below. Using the framework for growth of mathematical understanding (Pirie & Kieren, 1994) as referent, the learning activity was considered to be at the formalising layer. In this learning activity, learners were required to consciously notice properties related to the midpoint theorem and use them to calculate the interior angles of ΔMNT. Thus, the learning task required learners to reason with properties of the objects (Gibbons, 2012; Wright, 2014).

In Figure 3, the points M, N and T are the midpoints of sides AB, BC, and AC respectively in ΔABC. Calculate the interior angles of ΔMNT.

The learners seemed to notice that they could use the idea that the sum of angles in a triangle are equal to 180°. They were observed calculating the magnitude of $\angle$C. They also determined the magnitude of the other angles in the diagram before evaluating the interior angles of ΔMNT. Once they finished, they studied the diagram in silence until Sipho asked, 'So how do we find angles M, N and T?' The extract that follows represents the interactions that emerged:

2.1. Sipho: How do we determine angles M, N and T?

2.2. Lesiba: Is it not possible to use midpoint theorem?

2.3. Sipho: Isn't this F? This is F … you see …

2.4. Lebogang: Then it is corresponding angles ...

2.5. Lesiba: Of parallel lines.

2.6. Lebogang: It means that here [pointing to $\angle$N], we are going to represent it as $\angle N_1$, $\angle$N2 [T$\hat{N}$C], here [pointing to $\angle$T] $\angle T_1$ and $\angle T_2$.

2.7. Lesiba: But here they didn't give us $\angle$N1 and $\angle$N2. [Inaudible] $\angle$ of ΔMTN and then midpoint of AC is T … midpoint MN and TR are midpoints of...

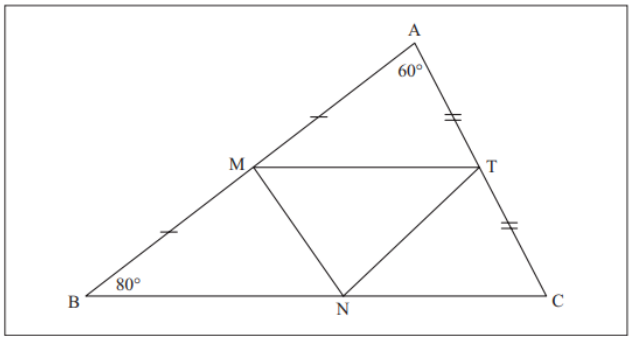


FIGURE 3: Application of the midpoint theorem.

2.8. Lebogang: If we can say $\angle N_1$ and $\angle N_2$ we will understand that $\angle B$ is equal to $\angle N_2$, then they are corresponding.

2.9. Sipho: Ohoo ... understand, these are the angles that Lebogang is talking about, this is F…

2.10. Lesiba: I can see that …

2.11. Lebogang: This means $\angle N_2$ is equal to 80°.

2.12. Sipho: Wait a minute! Oh yes I can see that.

2.13. Lebogang: Then here it's Δ TNC.

2.14. Sipho: Hmmm.

2.15. Lesiba: Hmmm.

2.16. Sipho: You are using a long way … Ohooo yes continue …

2.17. Lebogang: Then in Δ TNC, we are going to do just like the first part, we take $\angle N_2$ and add it with $\angle$ N then subtract from 180° to get $\angle$ T2 [NT̂C].

2.18. Sipho: Hmmm I see it.

2.19. Lesiba: We are supposed to find $\angle$ M, $\angle$ N and $\angle$ T.

2.20. Lebogang: Wait, listen, $\angle N_2$ is alternating to $\angle T_1$.

2.21. Sipho: Show [Pause] Yes, carry on Lebogang.

2.22. Lebogang: [starts writing]

2.23. Sipho: So $\angle T_1$ is 80°.

2.24. Lesiba: Alternating angles are equal …

2.25. Lebogang: Alternating angles are equal. It means that $\angle T_1$ is 80°.

The extract began with Sipho asking the group members how to get the size of the angles in triangle MTN. Lesiba suggested that they use the midpoint theorem, an indication that he was operating at the image making level. Pointing at TN̂C and MB̂N, Sipho examined the diagram and said that it satisfies the properties for F, a mnemonic that is used for corresponding angles that are formed from parallel lines. Even though the learners did not give a mathematically justifiable reason, this suggests that he was aware that line BM is parallel to line NT. The group followed on this train of thought by providing more relevant information and identified that TN̂C and MB̂N are corresponding angles formed by parallel lines. This exploratory talk assisted Lebogang to engage in acts of property noticing; she noticed that each of angles $\angle$ N and $\angle$ T consisted of three angles. She named them $\angle N_1$, $\angle N_2$, $\angle N_3$, and $\angle T_1$, $\angle T_2$, $\angle$ T (see Figure 4).

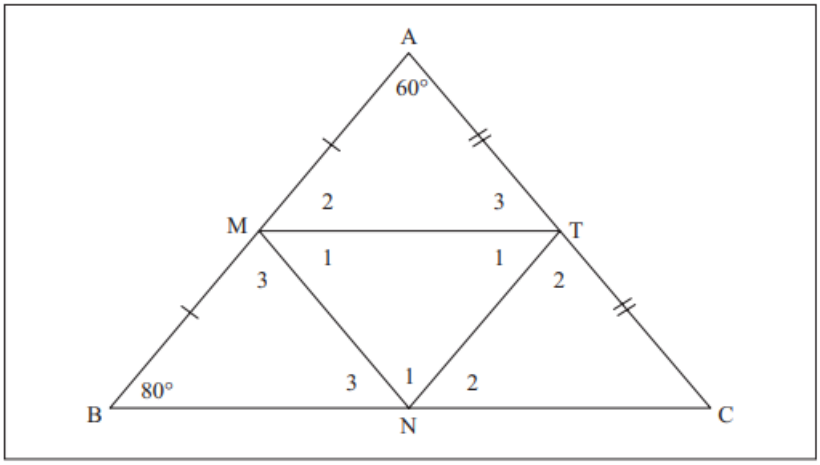


FIGURE 4: Rotation of a part of Figure 1 to clarify its properties

This further confirms Rojas-Drummond et al.'s (2013) and Mercer and Wegerif's (2004) assertions that reasoning is visible in exploratory talk.

It seemed Lesiba did not understand Lebogang's workings because he did not notice how that related with the midpoint theorem. But once he realised that she did this to show that $T\hat{N}C = \angle B = 80°$ where $T\hat{N}C = \angle N_2$ he accepted what Lebogang did. In this instance, there is evidence of exploratory talk which resulted in explicit peer intentional intervention, as observed where both Lebogang and Lesiba developed an understanding of the ideas used towards solving the given activity. Thus, through exploratory talk, Lesiba appeared to be actively involved in the resolution of the learning activity as he could share his ideas concerning the learning activity. Furthermore, Lesiba's utterances in lines 2.7 and 2.10 highlight instances where 'opinions are sought and considered before decisions are jointly made' (Mercer & Wegerif, 2004, p. 72) during exploratory talk.

It is also evident in the vignette that through exploratory talk learners had different ways of working on a mathematics task. In this extract, although Sipho perceived that Lebogang used a lengthy method to solve the problem, immediately after Sipho's utterance, it seems he realised the knowledge that Lebogang used towards solving the problem; he said: 'Ohooo yes, continue'. This affirms Barnes's (2010) concession that exploratory talk offers alternative ways of thinking about a task, which account for valued contributions in the learning environment (Chan, 2020). The utterance indicates that although Sipho almost challenged what Lebogang did, he was attentive. In this instance, we claim that Sipho developed an understanding of the approach used by Lebogang. Hence, Sipho's utterance in line 2.16 shows that what happened was now shared.

Lebogang proceeded to show that $N\hat{T}C$ is computed by subtracting the sum of $\angle C$ and $T\hat{N}C$ from 180°. Exploratory talk eased the learners to accept challenging ideas presented by their peers. This ease aligns itself with a characteristic of exploratory talk, where learners' ideas become challenged in a learning environment (Dahl et al., 2018; Mercer & Wegerif, 2004).

Although learners were able to compute the magnitude of other angles such as $N\hat{T}C$ with appropriate geometry reasoning, Lesiba made them aware that the computation did not solve the given problem because they were supposed to calculate the interior angles of ΔMNT. In this case folding back caused a discontinuity. It can be claimed that in this group the learners felt at ease with challenging ideas that are presented by others.

Lesiba's exploratory talk resulted in learners folding back from property noticing to the formalising layer where they applied properties of various angles to determine the interior of angles of ΔMNT, as follows:

2.26. Lebogang: Wait, listen, $\angle N_2$ is alternating to $\angle T_1$.

2.27. Sipho: Show [Pause] Yes, carry on Lebogang.

2.28. Lebogang: [starts writing] $\angle T_1 = \angle N_2$

2.29. Sipho: So $\angle T_1$ is 80°.

2.30. Lesiba: Alternating angles are equal ...

2.31. Lebogang: Alternating angles are equal. It means that $\angle T_1$ is 80°.

2.32. Lesiba: Ohoo so $\angle T_1$ is the angle inside ΔMNT. Here [pointing at $\angle M_1$] is equal to $\angle$ T3.

2.33. Lebogang: Yes Lesiba, here we can see that $\angle T_3$ is 40° because it is corresponding with $\angle C$.

2.34. Sipho: So then $\angle M_1$ is equal to 40°, these [$\angle T_3$ and $\angle M_1$] are alternating angles.

2.35. Lesiba: Yes.

2.36. Lebogang: Then we can use the sum of angles in ΔMNT to find $\angle N_1$.

2.37. Sipho: Yes, let's write it.

2.38. Lesiba: Okay I see it, $\angle T_1$ plus $\angle M_1$ plus $\angle$ N1 is 180°.

2.39. Sipho: Sum of angles in a triangle.

2.40. Lebogang: [starts writing] $\angle T_1 + \angle M_1 + \angle N_1 = 180°$ (Sum of $\angle$s in triangle).

2.41. Lesiba: Let us substitute the values of the other two angles.

2.42. Lebogang: [continues writing] $\angle N_1 = 180° – 40° – 80°$.

2.43. Sipho: Yes, so $\angle N_1 = 60°$.

The above extracts illustrated that Lebogang successfully demonstrated how $\angle T$ of ΔMNT is calculated, evidence that she consciously noticed properties of alternating angles and worked with them. In addition, Sipho added by giving the size of $\angle T$ and Lesiba justified why what they did is geometrically accurate – an indication that everyone can contribute during exploratory talk of problem-solving. Eventually the learners were able to fold back to the formalising level where they determined the magnitude of the angles of ΔMNT.

Conclusion and Recommendations

This article argues that folding back is successful when the learners engage in exploratory talk. We supported this argument by providing extracts of talk among groups of learners. These talks showed that learners could fold back as they worked on the given problems because they (1) allowed individuals to take the lead in the talk, (2) followed up on ideas presented in the group, and (3) either supported or challenged ideas in an uncompetitive way.

In the article, we pursued the question: What kind of talk is necessary for folding back to help learners develop geometry understanding? We used the narrative mode of data analysis to analyse the type of talk learners engage in while folding back (Martin, 2008). Two groups of learners were video recorded while working on geometry learning activities and our focus was on the learners' instances that promoted the building of ideas through folding back. The article shows that exploratory talk promotes folding back where learners build on each other's ideas to develop geometry understanding. As learners moved across different layers of understanding, they shared their thought processes which helped their peers to enhance their geometry understanding (Extract 1 and Extract 2). Banes et al. (2020) claimed that exploratory talk in mathematics classrooms is rare. Therefore, this claim necessitates studies that explore engendering exploratory talk in teaching and learning of geometry in schools. Furthermore, we encourage research to examine the understanding of geometry concepts using Pirie and Kieren's (1994) conception of folding back and Wegerif and Mercer's (2004) three ways of talking and thinking.

This finding concurs with Towers and Martin's (2014) study, which attests that individual learners' actions and statements contribute towards building understanding for a group of learners. In addition, the finding may be comparable to Hunter and Civil's (2021, p. 16) observation of learners who participate in collaborative groups, not as singles but 'as an interrelated and interdependent organism'. The finding was evident when learners explained and elaborated their actions, such as using mnemonics to solve the geometry learning activities. For Wei and Ismail (2010) and Martin and Towers (2015), this leads to a pathway for learners to modify and build their mathematical understanding. Through folding back, learners become interested not only in their geometry understanding but also in their peers' understanding. This results in an opportunity for peers to explain their preferred approach in solving geometry learning activities.

Furthermore, folding back promotes an environment where ideas are not only shared but appreciated. Through folding back, individual learners respect and encourage their peers to apply and articulate preferred methods to arrive at the solution of a geometry learning activity. Learners' ability to articulate their actions in resolving geometry learning activities not only promotes their confidence but plays a significant role in their development of geometry understanding (Mabotja, 2017; Yao & Manouchehri, 2022). As a result, learners become collaborative instead of competitive in their geometry learning environment. Our

overall reflection on the cited narratives prompts us to conclude that for teachers who wish to promote growth in understanding through folding back, the type of talk that should be normative is exploratory talk. Teachers should request learners to explain, evaluate, and argue, while trading off merely finding answers for classroom talk that emphasises reasoning (Røsseland et al., 2022).

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